# MATH 2060 BC TUTO 1

**Def.** Let
- $I \subseteq \mathbb{R}$ be an interval
- $f : I \to \mathbb{R}$ be a fcn on $I$
- $c \in I$

- The derivative of $f$ at $c$ is given by the limit
$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$
provided this limit exist.
- In this case, we say that $f$ is differentiable at $c$.

**Thm.** If $f : I \to \mathbb{R}$ has a derivative at $c \in I$ (i.e. differentiable at $c$), then $f$ is continuous at $c$.

**Example 1** ( §6.1 Ex 7 )

Suppose that $f$ is diff. at $c$ and that $f(c) = 0$.
Show that $g(x) := |f(x)|$ is diff. at $c$ iff $f'(c) = 0$.

Ans : $\forall x \neq c$,

$$\frac{g(x)-g(c)}{x-c} = \frac{|f(x)|-|f(c)|}{x-c}$$

$|f(x) - f(c)|$

$\operatorname{sgn}(x-c)\,|x-c|$

$$= \operatorname{sgn}(x-c)\left|\frac{f(x)-f(c)}{x-c}\right| \quad \text{since } f(c) = 0.$$

Here
$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x>0 \\ 0 & \text{if } x=0 \\ -1 & \text{if } x<0. \end{cases}$$

Then
$$\lim_{x\to c^+}\frac{g(x)-g(c)}{x-c} = \lim_{x\to c^+}\operatorname{sgn}(\underbrace{x-c}_{>0})\left|\frac{f(x)-f(c)}{x-c}\right| = |f'(c)|$$

$$\lim_{x\to c^-}\frac{g(x)-g(c)}{x-c} = \lim_{x\to c^-}\operatorname{sgn}(\underbrace{x-c}_{<0})\left|\frac{f(x)-f(c)}{x-c}\right| = -|f'(c)|$$

Note $\displaystyle\lim_{x\to c}\frac{g(x)-g(c)}{x-c}$ exists iff $\displaystyle\lim_{x\to c^+}\frac{g(x)-g(c)}{x-c} = \lim_{x\to c^-}\frac{g(x)-g(c)}{x-c}$

So $g$ is diff. at $c$ iff $|f'(c)| = -|f'(c)|$
iff $f'(c) = 0$.

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__Example 2__ (§6.1 Ex 10)

Let $g:\mathbb{R}\to\mathbb{R}$ be defined by

$$g(x)=\begin{cases} x^2\sin(1/x^2) & \text{if } x\neq 0 \\ 0 & \text{if } x=0. \end{cases}$$

Show that $g$ is diff. for all $x\in\mathbb{R}$.
Also show that $g'$ is not bounded on $[-1,1]$.

Ans: $x\neq 0$: By chain rule and product rule,

$$g'(x)=2x\sin(1/x^2)+x^2\cos(1/x^2)(-2x^{-3})$$
$$=2x\sin(1/x^2)-\frac{2}{x}\cos(1/x^2)$$

$x=0$: $$\lim_{x\to 0}\frac{g(x)-g(0)}{x-0}=\lim_{x\to 0}x\sin\left(\frac{1}{x^2}\right)=0$$

by Squeeze Thm $\left(\left|\sin\left(\frac{1}{x^2}\right)\right|\leq 1\ \forall x\neq 0\right)$

So $g'(x)$ exists $\forall x\in\mathbb{R}$, i.e. $g$ is diff. $\forall x\in\mathbb{R}$.

For unboundedness, wish to find $x_n\in[-1,1]$ s.t. $x_n\to 0$,

$$\sin\left(\frac{1}{x_n^2}\right)=0 \quad\text{and}\quad \cos\left(\frac{1}{x_n^2}\right)=1$$

Let $x_n:=\frac{1}{\sqrt{2n\pi}}$, $n\in\mathbb{N}$.

Then $\{x_n\}$ satisfies the desired properties and

$$|g'(x_n)|=\left|0-2\sqrt{2n\pi}\cdot 1\right|=2\sqrt{2n\pi}\to\infty. \text{ as } n\to\infty.$$

So $g'$ is NOT bounded on $[-1,1]$

## Example 3 (§6.1 Ex 12)

If $r > 0$ is a rational number, let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) := \begin{cases} x^r \sin\left(\frac{1}{x}\right) & , \quad x \neq 0 \\ 0 & , \quad x = 0 . \end{cases}$$

Determine those values of $r$ for which $f'(0)$ exists.

Ans: Note $\displaystyle\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} x^{r-1} \sin\left(\frac{1}{x}\right)$

Case 1: $r > 1$.

Then $\left|x^{r-1} \sin\left(\frac{1}{x}\right)\right| \leq |x|^{r-1} \quad \forall\, x \neq 0$

and $\displaystyle\lim_{x \to 0} |x|^{r-1} = 0$

By Squeeze Thm, $\displaystyle f'(0) = \lim_{x \to 0} x^{r-1} \sin\left(\frac{1}{x}\right) = 0$.

Case 2: $0 < r \leq 1$.

Take $x_n = \frac{1}{2n\pi}$. and $y_n = \frac{1}{2n\pi + \frac{\pi}{2}}$

Then $x_n, y_n \neq 0 \ \forall n$ and $x_n, y_n \to 0$

However, $x_n^{r-1} \sin\left(\frac{1}{x_n}\right) = 0$

while $y_n^{r-1} \sin\left(\frac{1}{y_n}\right) = \left(2n\pi + \frac{\pi}{2}\right)^{1-r} \begin{cases} = 1 & , \ r = 1 \\ \to \infty & , \ r < 1 \end{cases}$

By Sequential Criterion, $\displaystyle f'(0) = \lim_{x \to 0} x^{r-1} \sin\left(\frac{1}{x}\right)$ DNE.

Conclusion: $f'(0)$ exists iff $r > 1$ (and $r \in \mathbb{Q}$) //

**Example 4** (§6.1 Ex 13)

If $f: \mathbb{R} \to \mathbb{R}$ is diff. at $c \in \mathbb{R}$, show that
$$f'(c) = \lim \left( n \{ f(c + 1/n) - f(c) \} \right)$$
However, show by example that the existence of the limit of this seq. does not imply the existence of $f'(c)$

Ans: Since $f$ is diff. at $c$,
$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = f'(c)$$

Consider the seq $\{h_n\}$, $h_n := \frac{1}{n}$,

we have $h_n \neq 0$ and $\lim(h_n) = 0$

By Sequential Criterion for Limits of fcns,
$$\lim_{n \to \infty} \frac{f(c + h_n) - f(c)}{h_n} = f'(c)$$
$\frac{1}{n}$ $\frac{1}{n}$

i.e. $f'(c) = \lim_{n \to \infty} n \left[ f(c + \frac{1}{n}) - f(c) \right]$ //

For the counterexample, one may consider the Dirichlet fcn
$$f(x) := \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Then $n\left[f(c+\frac{1}{n}) - f(c)\right] = \begin{cases} n(1-1) & \text{if } c \in \mathbb{Q} \\ n(0-0) & \text{if } c \notin \mathbb{Q} \end{cases}$

$= 0 \qquad \forall n \in \mathbb{N}, \ \forall c \in \mathbb{R}.$

$\Rightarrow \lim \left( n\left[f(c+\frac{1}{n}) - f(c)\right] \right) = 0$

However, $f'(c)$ DNE for any $c \in \mathbb{R}$ since $f$ is discontinuous everywhere. //

**Example 5** Let $f:(-a,a)\to\mathbb{R}$ with $a>0$.
Assume $f$ is cts at $0$ and s.t. the limit

$$\lim_{x\to 0}\frac{f(x)-f(\lambda x)}{x}=\ell \qquad (*)$$

exists, where $0<\lambda<1$.

a) Show that $f'(0)$ exists

b) What happens to the conclusion when $\lambda>1$.

Ans: a) Let $\varepsilon>0$.

By $(*)$, $\exists\,\delta>0$ s.t. $\forall x\in(-a,a)$ with $0<|x|<\delta$, we have

$$\ell-\varepsilon<\frac{f(x)-f(\lambda x)}{x}<\ell+\varepsilon$$

Idea: $$\underbrace{f(x)-f(\lambda x)+f(\lambda x)-f(\lambda^2 x)+\cdots+f(\lambda^n x)-f(\lambda^{n+1}x)}_{=f(x)-f(\lambda^{n+1}x)}$$

$f(\lambda^{n+1}x)\to f(0)$ as $n\to\infty$

Let $x\in(-a,a)$ with $0<|x|<\delta$.

Then $0<|\lambda^n x|\leq|x|<\delta$ $\forall n\in\mathbb{N}\cup\{0\}$ since $\lambda\in(0,1)$,

Hence, $\forall n\in\mathbb{N}\cup\{0\}$,

$$\ell-\varepsilon<\frac{f(\lambda^n x)-f(\lambda\cdot\lambda^n x)}{\lambda^n x}<\ell+\varepsilon$$

$$(\ell-\varepsilon)\lambda^n<\frac{f(\lambda^n x)-f(\lambda\cdot\lambda^n x)}{x}<(\ell+\varepsilon)\lambda^n$$

So, $\forall N\geq 1$, $$\sum_{n=0}^{N}(\ell-\varepsilon)\lambda^n<\sum_{n=0}^{N}\frac{f(\lambda^n x)-f(\lambda\cdot\lambda^n x)}{x}<\sum_{n=0}^{N}(\ell+\varepsilon)\lambda^n$$

i.e. $$(\ell-\varepsilon)\frac{1-\lambda^{N+1}}{1-\lambda}<\frac{f(x)-f(\lambda^{N+1}x)}{x}<(\ell+\varepsilon)\frac{1-\lambda^{N+1}}{1-\lambda}$$

$f(0) = \lim_{N\to\infty} f(\lambda^{N+1}x)$ since $f$ is cts at $0$

Letting $N \to \infty$, we have

$$\frac{\ell - \varepsilon}{1-\lambda} \leqslant \frac{f(x)-f(0)}{x} \leqslant \frac{\ell+\varepsilon}{1-\lambda}$$

So, $$\left| \frac{f(x)-f(0)}{x} - \frac{\ell}{1-\lambda} \right| \leqslant \frac{\varepsilon}{1-\lambda}$$ whenever $0<|x|<\delta$.

Therefore $f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x} = \frac{\ell}{1-\lambda}$ .

b) If $\lambda > 1$, then $0 < \frac{1}{\lambda} < 1$.

By letting $t = \lambda x$, we have

$$\ell = \lim_{x\to 0} \frac{f(x)-f(\lambda x)}{x} = \lim_{t\to 0} \frac{f(\frac{t}{\lambda}) - f(t)}{\frac{t}{\lambda}} = -\lambda \lim_{t \to 0} \frac{f(t)-f(\frac{t}{\lambda})}{t}$$

and so $$\lim_{t\to 0} \frac{f(t)-f(\frac{t}{\lambda})}{t} = -\frac{\ell}{\lambda}.$$

By a), $f'(0)$ exists and

$$f'(0) = \frac{-\frac{\ell}{\lambda}}{1-\frac{1}{\lambda}} = \frac{\ell}{1-\lambda}.$$ (since $0<\frac{1}{\lambda}<1$)

Therefore $f'(0) = \frac{\ell}{1-\lambda}$ for $0<\lambda<1$, $\lambda>1$ //